Transition from Clumpy to Smooth Angular Diameter Distances

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ABSTRACT

Distance relations in a locally inhomogeneous universe are expected to behave like the Dyer-Roeder solution on small angular scales and the Friedmann-Robertson-Walker solution on large angular scales. Within a simple compact clump model the transition between these asymptotic behaviors is demonstrated and quantified. The redshift dependent transition scale is of order a few arcseconds; this implies it should have little influence on large angular scale cosmological tests such as the volume-redshift relation but possibly significant effects on arcsecond angular diameter measurements of radio galaxies and AGNs. For example, at $z = 2$ on arcsecond scales a clumpy flat universe mimics the angular diameter distance of a smooth $\Omega = 0.27$ model.

Subject headings: gravitational lensing, distance scale, cosmology: theory

1.   Introduction

Light propagation through our universe is generally treated in two separate regimes: either within the smooth Friedmann-Robertson-Walker model or within the gravitational lensing model of density inhomogeneities (see, e.g., Schneider, Ehlers, & Falco 1992). In the actual universe both local inhomogeneities and global homogeneity exist, so each acts as an asymptotic description under the appropriate conditions. Along with image amplifications, distortions, and time delays, the source distance-redshift relation is also affected by inhomogeneities. While many researchers discuss the distance relations in the presence of inhomogeneities, e.g. Futamase & Sasaki (1989), Watanabe & Tomita (1990), and how ignorance of clumpiness can affect determination of cosmological parameters by distance tests, e.g. Linder (1988a), Hadrović & Binney (1997), the question of the nature of the transition region between the two regimes is not addressed. It is of importance to understand where this transition occurs so as to use the two limiting cases only where they are valid approximations.

For almost all astrophysical applications of light propagation geometric optics holds, where the curvature or inhomogeneity scales are much greater than the wavelength of the electromagnetic radiation, so the radiation can be treated as a beam or bundle of light rays. The focusing of the bundle due to the spacetime geometry determines the angular diameter distance r by

$$d^2r/d\lambda^2 = -\left(\mathcal{R} + |\sigma|^2\right) r, \tag{1}$$

where λ is the affine parameter measuring the path length, $\mathcal{R} = (1/2)R_{\mu\nu}k^\mu k^\nu$ is the Ricci contribution of the gravitational focusing from matter within the beam, and σ is the Weyl contribution of gravitational shear from matter outside the beam. Here $R_{\mu\nu}$ is the Ricci tensor and k^μ the photon four momentum.

To obtain a universal, i.e. isotropic, distance relation, equation (1) is generally solved under the "average path" assumption (Dyer & Roeder 1973) for an infinitesimal light beam, e.g. a single ray. This posits that the matter density along the line of sight is given by the global average density times a uniform smoothness parameter α, measuring the degree of small scale inhomogeneity or clumpiness, and that the shear vanishes from global homogeneity. This approximation substitutes for detailed knowledge of the locally inhomogeneous metric, or physical conditions, along the light path, which we generally lack. Given that ignorance some such effective model must be adopted, with the main caution being insurance that a "typical" path is indeed characteristic of the average.

As the light beam subtends larger and larger solid angles, at some point the inhomogeneities should be smoothed over and we can legitimately calculate distances within the Friedmann-Robertson-Walker cosmology. The question arises how to connect these two asymptotic behaviors with a reasonable, preferably simple analytic model. (One can of course use numerical ray shooting within a specific pattern of inhomogeneities to compute the distance relations but they will be relation*s*, not one single universal relation as desired.) Of particular observational interest is how large is the transition angle, as many cosmological tests such as the number-redshift and magnitude-redshift relations are sensitive to the precise distance measure.

2.   Smoothing

The standard model for density inhomogeneities when calculating cosmological distance relations is the Dyer-Roeder prescription: distribute randomly a fraction $1 - \alpha$ of the total matter density in clumps, point masses M with constant comoving number density

n_0. Light rays passing too close to a clump will be appreciably gravitationally influenced (spatially deflected, cross sectionally sheared, intensity amplified) and so will be recognized as something special and not treated as being on a typical line of sight. That leaves those paths avoiding clumps, but their neighborhoods only possess a matter density of $\alpha\rho$ where ρ is the global average matter density. Thus, an "average" path does not feel the full Friedmann-Robertson-Walker (FRW) density and the calculated angular diameter distance $r(\alpha, z)$ will differ from the FRW relation $r_{FRW} = r(1, z)$. In this case $\mathcal{R} = 4\pi(1 + z)^2\alpha\rho$ where z is the source redshift.

[One can also generalize the model to redshift dependent clumpiness $\alpha(z)$, modeling the effects of evolving inhomogeneity, clump mass, or number density (Linder 1988b). Note that this brings up an interesting point. If we lived in a local inhomogeneity, either a bubble of different density or different clumpiness, then even though the volume averaged universe would appear Friedmann, our observed distance-redshift relation would not be the same as in that volume averaged universe. It does not asymptotically approach the Friedmann result even at distances much greater than the bubble size. I.e. the observer is in a special position and volume rather than angle (line of sight) averaging may give misleading results; one cannot invoke an ergodic homogeneity because our position as observers selects a unique location. Such an isotropic inhomogeneity would lead to an apparent conflict between cosmological parameters derived from dynamical quantities and those involving the distance-redshift relation (in preparation).]

Given the Dyer-Roeder ansatz, one can see that as a light beam subtends a larger solid angle, the probability increases that it will include a clump. This raises the matter density within the beam and when the bundle is broad enough it will feel the full FRW density. To investigate this transition consider a beam with half angle θ at the observer, propagating from a redshift z. Let each clump own a conical volume V_c given by

$$\int n\, dV_c = 1, \tag{2}$$

which extends from the observer out to the source or survey depth z with half angle θ_c. Here n is the proper number density; note the subscript c on the proper volume element dV denotes "clump" not "comoving".

The probability that the light beam will include the clump is given by the Poisson process

$$p(\theta) = 1 - e^{-\tau(\theta)}, \tag{3}$$

where the optical depth $\tau = V(\theta, z)/V(\theta_c(z))$. That is, on average a beam covering V_c

has roughly unity probability for including the clump (really $1 - e^{-1}$ because of statistical fluctuations in the number density).

Although as the beam expands, i.e. θ increases, the matter within the beam actually jumps in a step function, as the next step in our ansatz we simulate the global averaging by choosing an effective smoothed density $\rho_{eff} = \bar{\alpha}\rho + \rho(1 - \bar{\alpha})\,p(\theta)$, where $\bar{\alpha}$ is the global smoothness parameter, i.e. that felt by an infinitesimal ray. So rather than having no clump within the beam then suddenly all of it, we smoothly add its contribution according to the probability for inclusion $p(\theta)$. This is the main approximation we make in order to talk about a universal distance function. It seems physically reasonable and is well behaved mathematically.

The effective smoothness parameter is now

$$\alpha(\theta, z) = \bar{\alpha} + (1 - \bar{\alpha})\Big[1 - \exp\{-V(\theta, z)/V[\theta_c(z)]\}\Big]. \tag{4}$$

The volumes are given by

$$\begin{aligned} V(\theta, z) &= \int_z dr_p \int_\theta dA = \int_z dr_p \int_\theta d\omega\, r^2[\alpha(\vartheta)] \\ &= 2\pi \int_z dr_p \int_0^\theta d\vartheta\, \vartheta r^2[\alpha(\vartheta)] \equiv \pi\theta^2 g(z, \alpha), \end{aligned} \tag{5}$$

where r_p is the proper distance along the line of sight, A is the transverse area on the sky, and ω the angular area on the sky.

Substituting equation (5) into (4) yields

$$\begin{aligned} \alpha(\theta) &= \bar{\alpha} + (1 - \bar{\alpha})\Big[1 - \exp\{-(\theta/\theta_c)^2[g(z, \alpha)/g(z, \alpha_c)]\}\Big], &(6a) \\ \alpha_{apx}(\theta) &\equiv \bar{\alpha} + (1 - \bar{\alpha})[1 - \exp\{-(\theta/\theta_c)^2\}], &(6b) \end{aligned}$$

plotted in Figure 1, where $\alpha_c \equiv \alpha(\theta_c) = 1 - e^{-1}(1 - \bar{\alpha})$. We adopt $\Omega = 1$, which would cause the greatest deviation of the clumpy vs. smooth FRW model. Note that the approximation in (6b) (solid curve) has removed the need for recursion in determining $\alpha(\theta)$. Because the ratio of g's is insensitive to α the approximation is excellent: the fully recursed α differs from α_{apx} by at most 11% for the extreme case of $\bar{\alpha} = 0$, $z = 3$ (and is more typically $< 1\%$) – and this occurs in a region of parameter space such that the distance $r(\alpha)$ is never more than 1% from $r(\alpha_{apx})$.

The resulting $\alpha(\theta)$ has the pleasing properties of monotonicity, simplicity, and the proper asymptotic behaviors: as θ increases the parameter goes from $\alpha(\theta \ll \theta_c) = \bar{\alpha}$ to $\alpha(\theta \gg \theta_c) = 1$, i.e. we have effectively introduced an averaging procedure that provides the Dyer-Roeder clumpy universe result at small angles, the FRW result at large angles, and defines the transition.

To find the transition angle θ_c combine equations (2) and (5) to get

$$1 = 2\pi n_0 \int_0^{\theta_c} d\theta \, \theta \int_z dr_p \, (1+z)^3 r^2 [\alpha(\theta)] \equiv \pi \theta_c^2 n_0 H_0^{-3} f(z, \alpha_c), \tag{7}$$

for constant comoving clump density. So

$$\theta_c = [\pi n_0 H_0^{-3} f(z, \alpha_c)]^{-1/2}. \tag{8}$$

Taking $n_0 = (3H_0^2/8\pi)\Omega_c M^{-1}$ where $H_0 = 100 \, h$ km s^{-1} Mpc^{-1} is the Hubble constant and $\Omega_c = (1 - \bar{\alpha})\Omega$ the clump matter density in units of the FRW critical density,

$$\theta_c = 1.3'' \, (M/10^{12} M_\odot)^{1/2} h^{1/2} (1 - \bar{\alpha})^{-1/2} f^{-1/2}. \tag{9}$$

At low z or high smoothness $1 - \bar{\alpha} \ll 1$ the clump angle formally becomes large but this is unphysical given our interpretation of θ_c as a transition angle to FRW behavior. A more physical transition angle Θ is defined as that angle where $r(\alpha(\theta))$ first differs from r_{FRW} by less than 5%.

Figure 2 plots both Θ and θ_c vs. z. For low z the distances r are insensitive to α (it enters at third order in an expansion of r in z; see Linder 1988b) so even for $\theta = 0$ all $r(\alpha)$ are close to r_{FRW}. For high redshifts θ_c and Θ formally diverge as $\bar{\alpha} \to 1$ due respectively to the scarcity of clumps and to the difference in high z asymptotic behavior between smooth and even slightly clumpy distances. The growth is extremely slow, however, $\theta_c(z \gg 1) \approx 25''$ for $\bar{\alpha} = 0.999$ and $\Theta(z = 10^3) \approx 7''$ for $\bar{\alpha} = 0.99$. Realistically, $\bar{\alpha}$ evolves due to structure formation such that it approaches unity sufficiently closely (e.g. > 0.998 for $z \leq 10^3$) that $\Theta \to 0$ at high redshifts. Thus, in general distant source observations with fields of view larger than a few arcseconds can legitimately be treated within the FRW model for distances.

As a definite example, consider the volume-redshift cosmological test which involves the angular diameter distance (note the luminosity and proper motion distances are simply related to the angular diameter distance by factors of $1 + z$). From equation (5) the volume-redshift relation is

$$V(\theta) = V_{FRW}(\theta)[g(z,\alpha)/g(z,1)], \tag{10}$$

where $\alpha_{FRW} = 1$. For $z < 1$ the ratio of g factors is close to unity, and for $z > 1$ we observe at $\theta \gg \theta_c$ (larger than arcsecond fields) and so $\alpha(\theta) \approx 1$ and again $V \approx V_{FRW}$. Thus clumpiness effects are negligible in this case.

The differential volume test is more sensitive since this does not possess the integration in equation (5) that includes those very low redshifts where r is nearly independent of α (as discussed above). Then

$$\Delta V(\theta)/\Delta V_{FRW}(\theta) = 2\theta^{-2} \int_0^\theta d\vartheta \, \vartheta r^2[\alpha(\vartheta)]/r^2(1). \tag{11}$$

The ratio is bounded between $[r(\alpha(\theta))/r(1)]^2$ and $[r(0)/r(1)]^2$ and can approach 1.25 at $z = 1$ as $\theta \to 0$ if $\alpha(\theta)$ were unreasonably pushed to zero. The deviation from unity is less than 5% for all redshifts and clumpiness factors, however, when $\theta > 12''$. In particular, this assures us that the usual observations can be analyzed properly using the FRW volume element: for example the Loh-Spillar (1986) fields were $7' \times 10'$ at $z < 0.75$. (See Omote & Yoshida 1990 for how the clumpy luminosity distance affects the analysis of flux weighted counts.)

Clumpy and transition regimes can be important, however, for cosmological tests involving arcsecond scales. Possible applications include observations of radio galaxy lobes (Kapahi 1989, Guerra & Daly 1996) and milliarcsecond observations of active galactic nuclei (Gurvits 1994) for use in the angular diameter distance-redshift cosmological test. At a redshift of two, for example, the difference between the clumpy and smooth angular diameter distances for $\Omega = 1$ is 33%, and a clumpy flat model gives the same distance as a smooth model with $\Omega = 0.27$.

Figure 3 shows this trade off between clumpiness in a $\Omega = 1$ model and low density in a smooth model. Thus a flat but clumpy universe could be misinterpreted through small angular scale observations as a lower density one. In general one can match a distance in a (α,Ω) universe with a $(\alpha' > \alpha, \Omega' < \Omega)$ model. The numbers on the plotted curves give the angular scale of the observations in arcseconds and show the transition in clumpiness or alternatively density miscalculation from the infinitesimal case (Dyer-Roeder) to the large scale average (FRW).

3.   Shear

One point remains of which we must be cautious. The $\mathcal{R}$ term in equation (1) was treated by using an effective smoothed density but we neglected the shear term $|\sigma|^2$. On large scales we expect the shear to average to zero due to homogeneity but this is not ensured at smaller scales: if the light passes far from the clump then the shear on the bundle should be small, but not if it passes near the clump.

We can analyze this in terms of an effective clumpiness defined through equation (1) by

$$\alpha_{eff} = \alpha - (2/3)|\sigma|^2 H_0^{-2} \Omega^{-1} (1+z)^{-5}, \tag{12}$$

where Ω is the ratio of the total density to the critical density. Then the right hand side of equation (1) can be written simply as $4\pi(1+z)^2\alpha_{eff}\rho r$ and the previous results hold with the substitution of α_{eff} for α. Note that shear always decreases α_{eff} – brings it further from the FRW value of unity – and so increases the transition angle θ_c (more properly decreases the ratio θ/θ_c for which α_{eff} is a given value). Also note that α_{eff} can be negative, which causes no mathematical worries [it merely makes the usual parameter $\beta \equiv (25 - 24\alpha)^{1/2} > 5$].

Alternately, we can solve the full beam equation (1). Because the shear squared source term of a point mass dies off as $(\text{distance})^{-4}$, we can approximate its behavior as localized in affine parameter, say between λ_0 and $\lambda_0 + \Delta\lambda$. Dividing the light propagation from source at λ_s to observer at 0 into three regimes $(0, \lambda_0)$, $(\lambda_0, \lambda_0 + \Delta\lambda)$, $(\lambda_0 + \Delta\lambda, \lambda_s)$, we match the values of r and its first derivative at the boundaries. For the most extreme case, $\alpha = 0$ and $\Omega = 1$, the solutions to the distance relations are: $r_1 = \lambda$; $r_2 = C \sin|\sigma|\lambda + D \cos|\sigma|\lambda$; $r_3 = A\lambda + B$. Taylor expanding under the assumption $\Delta\lambda \ll \lambda_0, |\sigma|^{-1}$ yields

$$\begin{aligned} r_3(\lambda) &\approx & \lambda - |\sigma|^2\lambda_0^2\,\Delta\lambda(\lambda/\lambda_0 - 1) \\ &\approx & r_1(\lambda) - (4/25)|\sigma|^2\Delta z\, y_0^{-6}(1 - y_0^{-5/2})[1 - (y_0/y)^{5/2}], \end{aligned} \tag{13}$$

where $y = 1 + z$ and $\lambda = (2/5)(1 - y^{-5/2})$.

Thus the shear only has a significant effect on the angular diameter distance relation if the second term is nonnegligible. The maximum shear on the beam is given in order of magnitude by $|\sigma| \sim Mb^{-2}$ where b is the impact parameter from the mass M. For b equal to the Einstein radius of the mass, $|\sigma| \sim H_0(r_s/r_l r_{ls})$ where the r's are respectively measured from observer to source, observer to lens mass, and lens to source. The maximum value of the ratio of the second to first terms in equation (13) is then $(H_0\lambda_0)^2\Delta\lambda/\lambda \ll 1$, meaning

that when concerned with distances we can neglect shear for beams passing outside the Einstein ring of the mass.

So we argue that we can neglect the influence of shear on distances overall because 1) narrow beams ($\theta \ll \theta_E = b_E/r_l$) would typically miss the Einstein ring because their probability for intersection (θ_E^2/θ_c^2) is small (e.g. $\theta_E^2/\theta_c^2 \approx \Omega_c z^2 \ll 1$ for $z \ll 1$); 2) those few appreciably sheared light beams would be recognized as atypical and not used for distance measures; and 3) broad beams would have any shear effects diluted due to those portions of the beam that lie far from the clump and those symmetric (isotropic) about it. Therefore we do not expect shear to alter significantly the distance-redshift relation for a typical light bundle, except possibly at high redshifts where the transition scale θ_c is below an arcsecond (and so $\theta_c < \theta_E$) and hence below the region of observational interest.

4.   Conclusion

Using a simple toy model of an effective density distribution due to inhomogeneities one can derive a universal angular diameter distance relation applicable over all angular scales. It has the desired asymptotic properties: agreeing for small solid angles with the clumpy universe Dyer-Roeder distance, recreating for large solid angle observations the Friedmann-Robertson-Walker relation, and interpolating smoothly between them. The transition angle depends on the type of inhomogeneities but is estimated to be unlikely to exceed $10''$ for any cosmological test and should be of order $1''$ for any quantities involving the entire light propagation path between the source and observer. Still, this can have significant effects on such relations as the angular diameter-redshift relation for radio galaxies and AGNs, for example a true clumpy flat universe mimicking a smooth open model, and must be looked for in such cosmological tests.

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Fig. 1.— The effective smoothness α is plotted vs. beam size θ in units of the clump angle θ_c. The lower curves are for a completely clumpy universe, $\bar{\alpha} = 0$, while the upper have $\bar{\alpha} = 0.5$. Solid lines illustrate α_{apx} from (6b), dotted and dashed lines use the full α from (6a), evaluated at survey depths of $z = 1$ and $z = 3$ respectively.

Fig. 2.— The transition angle Θ (solid curves) and clump angle θ_c (dotted curves) in arcseconds are plotted vs. redshift. The pairs of curves are labeled near their intersections by the clumpiness $\bar{\alpha}$. At low redshifts the distance relation is close to the FRW behavior for all beam sizes while at high z it makes a transition to the clumpy behavior for angles smaller than Θ.

Fig. 3.— The clumpiness α needed for a flat universe to have the same angular diameter distance to redshift z as a smooth $\Omega < 1$ universe is plotted. In the conventional Dyer-Roeder model, for example, at $z = 2$ one could match $r(z)$ for $\alpha = 0$, $\Omega = 1$ by a FRW model with $\Omega = 0.27$. In the transition model of this paper, though, α is effectively a function of beam size. The numbers superposed on the curves give this size in arcseconds (for infinitesimal beam clumpiness $\bar{\alpha} = 0$). Thus observations with a $2''$ beam at $z = 2$ in a clumpy flat universe can be misinterpreted as belonging to a $\Omega = 0.54$ FRW universe.